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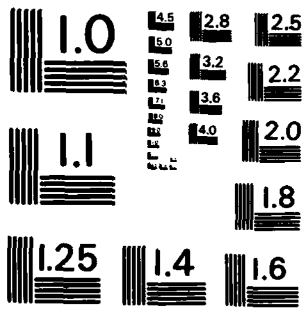

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A DIFFUSION EQUATION WITH A NONMONOTONE CONSTITUTIVE FUNCTION

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ABSTRACT

We discuss the well-posedness of the model problem

$$u_t = \phi(u_x)_x \quad \text{on} \quad [0,1] \times [0,T],\ T > 0\ , \qquad \text{(P)}$$

subject to given Neumann or Dirichlet boundary conditions at $x = 0$ and $x = 1$, and to the initial condition $u(x,0) = f(x)$; the given functions $f : [0,1] \to \mathbb{R}$, $\phi : \mathbb{R} \to \mathbb{R}$ are assumed to be smooth, $\phi(0) = 0$, ϕ satisfies the coercivity assumption $\xi\phi(\xi) \geq c\xi^2$, for some constant $c > 0$ and for $\xi \in \mathbb{R}$, and ϕ is assumed to be decreasing on an interval (a,b) with $a > 0$. We present a recent nonuniqueness result in the special case when ϕ is piecewise linear and study a related convexified problem.

AMS (MOS) Subject Classifications: 35K55, 35K65

Key Words: diffusion equation, nonlinear, nonmonotone constitutive function, a priori estimates, maximum principle, existence, nonuniqueness, convexification

Work Unit Number 1 - Applied Analysis

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SIGNIFICANCE AND EXPLANATION

The equation (P) in the abstract can be viewed as a simple model for nonlinear diffusion. The existing mathematical theory requires $\phi' > 0$. However, in one space dimension the laws of thermodynamics merely imply that the graph of ϕ lies in the first and third quadrant, without necessarily requiring ϕ to be monotone nondecreasing. This raises the natural question whether the assumption $\phi' > 0$ can be replaced by the much weaker coercivity condition $\xi\phi(\xi) > c\xi^2$, $c > 0$. For a nonmonotone, piecewise linear, coercive ϕ it was shown in MRC TSR #2354 (see [5]) that the initial value problem for (P) has infinitely many solutions, whenever the initial data $f'(\cdot)$ reach the critical range where $\phi'(\cdot) < 0$. These solutions u of (P) have the property that u_x oscillates between regions in which $\phi'(\cdot) > 0$. Although (P) is evidently not well-posed, it is hoped that imposing additional physically motivated assumptions will lead to a natural selection and a well-posed problem.

In this report we first review known (previously unpublished) a priori estimates for (P), and we then give a simpler construction for the existence of infinitely many solutions of (P) for a piecewise linear PHI ϕ as suggested by G. Strang [[illegible]]. We then investigate further the qualitative behavior of solutions of (P). Motivated by known results in one space dimension for the steady state, non-elliptic problem, $\phi(u_x)_x = 0$, we study the analogous convexified problem associated with (P). Analytical and numerical considerations suggest that the unique solution of the convexified problem (which has a monotone, nondecreasing constitutive function) can be interpreted as an average of solutions of (P), whenever the data $f'(\cdot)$ reach the critical range.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

A DIFFUSION EQUATION WITH A NONMONOTONE CONSTITUTIVE FUNCTION

Klaus Höllig[1,2] and John A. Nohel[1]

1. INTRODUCTION

We discuss the initial boundary value problem

$$
\begin{aligned}
&u_t = \phi(u_x)_x \quad \text{on} \quad [0,1] \times [0,T] \\
&u_x(0,t) = u_x(1,t) = 0, \quad u(x,0) = f(x)\ ,
\end{aligned}
\tag{P}
$$

where subscripts denote partial derivatives, under the principal assumption:

$$
\begin{aligned}
&\phi : \mathbb{R} \to \mathbb{R} \ \text{ smooth,} \quad \phi(0) = 0, \quad \text{and there exists a} \\
&\text{constant} \quad c > 0 \quad \text{such that} \quad \xi\phi(\xi) \geq c\xi^2, \ \xi \in \mathbb{R}, \\
&\text{and there exists an interval} \quad (a,b) \quad \text{with} \quad a > 0 \\
&\text{such that} \quad \phi'(\xi) < 0, \ \xi \in (a,b).
\end{aligned}
\tag{A}
$$

A model case is sketched in Figure 1.

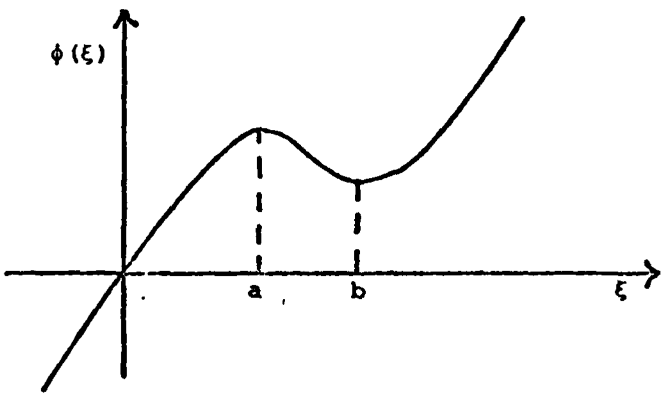

Figure 1

Concerning f we assume throughout that $f : [0,1] \to \mathbb{R}$ is as smooth as needed and satisfies the boundary conditions. Dirichlet and inhomogeneous

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boundary conditions can also be discussed, including the theorems in Sections 3 and 4.

One motivation for studying (P) is the theory of nonlinear diffusion. Specifically the Clausius-Duhem inequality [3, p. 79] in one space dimension can be shown to imply that the flux $\phi(u_x)$ need not be a monotone function of the gradient of temperature u_x; this is consistent with our assumption (A). If ϕ is strictly monotone increasing (as is usually assumed) standard theory guarantees that (P) has a unique solution which is, roughly speaking, as smooth as the function ϕ. In particular (see Figure 1), this is true in the model case if the data f satisfy $f'(x) < a$. If, however, $\phi'(f'(x_0)) < 0$ for some $x_0 \in (0,1)$, then in a neighborhood of $(x_0, 0)$, (P) behaves like a backward parabolic equation which is not well posed. In particular, if in Figure 1 $\phi(\xi) = \frac{1}{3}\xi^3 - \frac{3}{2}\xi^2 + 2\xi$, then (P) cannot have a solution u such that u_x is piecewise continuous on $[0,1] \times [0,T]$ for any $T > 0$, unless f is analytic. The basic problem to be discussed here is whether (P) is well-posed in some precise sense, whenever the data $f'(x)$ fall in the critical range ((a,b) in the model case).

The a priori estimates given in Section 2 suggest that (P) may have solutions u with $u_x \in L^\infty([0,1] \times [0,T])$ for some $T > 0$. Such solutions are constructed in Section 3 for piecewise linear ϕ satisfying assumption (A); however, the solutions are not unique. Motivated by known results for the steady-state, non-elliptic problem ($\phi(u_x)_x = 0$), we are lead to discussing in Section 4 the "convexified" problem associated with (P), with the objective (not yet established) of distinguishing a solution of (P) with special properties. Other open questions include the existence of solutions of (P) when ϕ is not piecewise linear, and a justification of phase changes suggested by numerical experiments.

2. REMARKS ON A PRIORI ESTIMATES

Consider (P) with either homogeneous Neumann or Dirichlet boundary conditions at $x = 0,1$, and with f smooth. Let $u \in W^{1,2}$ be a solution of (P) on $[0,1] \times [0,T]$ for some $T > 0$. Standard energy methods yield the following a priori information.

a. Multiplying by u and integrating the p.d.e. over $[0,1] \times [0,T]$ gives

$$\text{(i)} \quad \operatorname*{ess\,sup}_{t \in [0,T]} \int_0^1 u^2(x,t)\,dx \le \frac{1}{2} \int_0^1 f^2(x)\,dx, \quad \text{and}$$

$$\text{(ii)} \quad \int_0^T \int_0^1 u_x^2(x,t)\,dx\,dt \le \frac{1}{2c} \int_0^1 f^2(x)\,dx \ ,$$

where c is the coercivity constant in (A); if u satisfies Dirichlet b.c., the Poincaré inequality applied to (ii) also yields

$$\int_0^T \int_0^1 u^2\,dx\,dt \le \text{const.}$$

b. Multiplying the p.d.e. by u_t and integrating over $[0,1] \times [0,T]$ gives

$$\text{(i)} \quad \int_0^T \int_0^1 u_t^2\,dx\,dt \le \int_0^1 \Phi(f'(x))\,dx \ ,$$

where $\Phi'(\cdot) = \phi(\cdot)$ and

$$\text{(ii)} \quad \operatorname*{ess\,sup}_{t \in [0,T]} \int_0^1 \Phi(u_x(x,t))\,dx \le \int_0^1 \Phi(f'(x))\,dx$$

which, in view of the coercivity assumption, implies

$$\text{(iii)} \quad \operatorname*{ess\,sup}_{t \in [0,T]} \int_0^1 u_x^2(x,t)\,dx \le \frac{2}{c} \int_0^1 \Phi(f'(x))\,dx \ .$$

It should be observed that if ϕ is strictly monotone these estimates imply that $u \in H^2$, and standard techniques yield existence, without applying e.g. the theory of maximum monotone operators (in particular, this is true if $f'(x) < a$, see Figure 1). However, one cannot do this if ϕ is not monotone everywhere ($\phi'(f'(x)) < 0$ for some $x \in \mathbb{R}$).

A more subtle estimate, proved by J. Bona, L. Wahlbin and the second author, is

c. For either the Dirichlet or Neumann problem

$$\|u_x\|_\infty \leq \frac{1}{c} \sup_{x \in [0,1]} |\phi(f'(x))| ,$$

the optimal choice of the coercivity constant $c = \inf\limits_{y \in \mathbb{R}} \frac{\phi(y)}{y} > 0$.

Proof: Multiply the p.d.e. by $\phi^{2k}(u_x)u_t$, where in the exponent k is a positive integer, and integrate with respect to x and t. Defining

$$\Psi_k(y) = \int_0^y \phi^{2k+1}(\xi)d\xi \; (\geq 0 \text{ by hypothesis on } \phi \text{ for } y \in \mathbb{R})$$

one obtains, after using integration by parts and the boundary conditions:

$$\int_0^t \int_0^1 \phi^{2k}(u_x)u_t^2 dxdt + \frac{1}{2k+1} \int_0^1 \Psi_k(u_x)dx = \frac{1}{2k+1} \int_0^1 \Psi_k(f'(x))dx \; . \qquad (1)$$

From the coercivity assumption

$$\Psi_k(y) \geq \frac{c^{2k+1}}{2k+2} y^{2k+2} \; (y \in \mathbb{R}), \text{ and}$$

$$\int_0^1 \Psi_k(u_x)dx \geq \frac{c^{2k+1}}{2k+2} \int_0^1 u_x^{2k+2}dx \; .$$

Next let $M = \|f'\|_\infty$, $N = \sup_{x\in[0,1]} |\phi(f'(x))|$; then

$$\int_0^1 \Psi_k(f'(x))dx \le MN^{2k+1} .$$

Dropping the positive first term in (1), and then raising both sides to the $(2k+2)^{-1}$ yields

$$\frac{c^{\frac{2k+1}{2k+2}}}{(2k+2)^{\frac{1}{2k+2}}} \|u_x(\cdot,t)\|_{(2k+2)} \le M^{\frac{1}{2k+2}} N^{\frac{2k+1}{2k+2}}, \quad 0 \le t \le T .$$

Finally, letting $k \to \infty$ we obtain

$$\|u_x\|_\infty \le \frac{N}{c}, \quad 0 \le t \le T .$$

d. Maximum-minimum principle. Consider (P) in the model case of Figure 1 with homogeneous Neumann boundary conditions. Let $u \in W^{2,2}$ be a solution on $[0,1] \times [0,T]$ for some $T > 0$. Then for $0 \le t \le T$:

$$\min(a, \min_{0\le x\le 1} f'(x)) \le u_x(x,t) \le \max(b, \max_{0\le x\le 1} f'(x)) .$$

For classical solutions the result can be proved by a standard comparison method. We sketch a more general approach. Let $v = u_x$. Then v is a solution of the Dirichlet problem

$$\begin{aligned} v_t &= \phi(v)_{xx} \quad \text{on} \quad [0,1] \times [0,T] \\ v(x,0) &= f'(x); \quad v(0,t) = v(1,t) = 0 . \end{aligned} \tag{2}$$

Let $0 < a' < a$, $b' > b > 0$ (see Figure 1). Define $g : \mathbb{R} \to \mathbb{R}$ such that $g'(\cdot) = 0$ on (a',b') and $g'(\cdot) > 0$ otherwise. Let $G(s) = \int_0^s g(\xi)d\xi$. Multiply (1) by $g(v)$ and integrate over $[0,1] \times [0,t]$, $0 < t \le T$. Integration by parts and the boundary conditions yield

$$\int_0^1 G(v(x,t))dx + \int_0^t\int_0^1 \phi'(v)g'(v)v_x^2 dxdt = \int_0^1 G(f'(x))dx .$$

By definition of g and assumptions on ϕ the double integral is positive, so that

$$\int_0^1 G(v(x,t))dx \le \int_0^1 G(f'(x))dx, \quad 0 \le t \le T .$$

The successive specific choices

$$g(\cdot) = (\cdot - k)_+, \quad k = \max(b', \max_{0 \le x \le 1} f'(x)) ,$$

and

$$g(\cdot) = -(\tilde{k} - \cdot)_+, \quad \tilde{k} = \min(a', \min_{0 \le x \le 1} f'(x))$$

yield $G(v(x,0)) = G(f'(x)) = 0$. Therefore, respectively,

$$\frac{1}{2}\int_0^1 (v(x,t) - k)_+^2 dx \le 0, \quad 0 \le t \le T ,$$

and

$$\frac{1}{2}\int_0^1 (\tilde{k} - v(x,t))_+^2 dx \le 0, \quad 0 \le t \le T .$$

Thus

$$\tilde{k} \le v(x,t) \le k, \quad 0 \le t \le T .$$

Letting $a' \to a$ and $b' \to b$ yields the result. It should be noted that this stronger result, proved under the stronger assumption $u \in W^{2,2}$ (rather than $W^{1,2}$), is not implied by the L^∞ estimate in 2<u>c</u>.

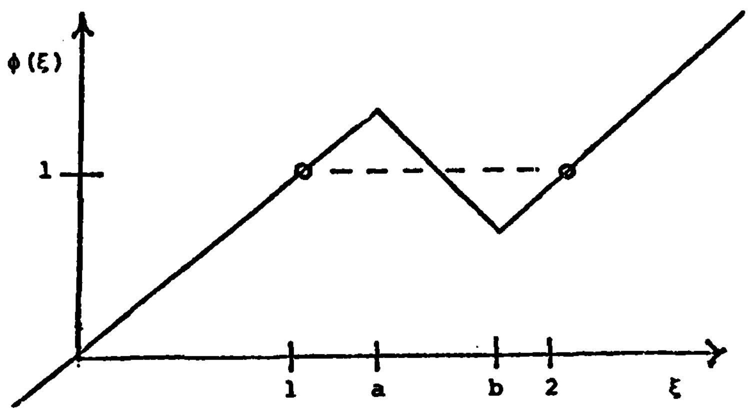
3. EXISTENCE AND NONUNIQUENESS IN A SPECIAL CASE

For a piecewise linear constitutive function ϕ it was shown in [5] that the problem (P) has infinitely many solutions if

$$\{x = \phi'(f'(x)) < 0\} \neq \emptyset : \tag{3}$$

<u>Theorem</u>. Let ϕ be piecewise linear with $\phi'((-\infty,a) \cup (b,\infty)) > 0$ and $\phi'((a,b)) < 0$ (see Figure 1) and assume that $f'(x)$ satisfies (3). There exists $T > 0$ such that (P) has infinitely many solutions u with $\phi(u_x)$ Hölder continuous, and $u_x \in L^\infty$, $u_t \in L^p$ $(p < \infty)$ on the domain $[0,1] \times [0,T]$. Moreover, $u_x(x,t) \notin [a,b]$ *for* $(x,t) \in [0,1] \times (0,T]$.

We outline the proof of the Theorem in the case when the increasing parts of ϕ are parallel, i.e. ϕ is of the form:



<u>Figure 2</u>

with $\phi'(\mathbb{R}\setminus[a,b]) = 1$. For simplicity we also assume that $f'(x) < 2$, $x \in [0,1]$. The key to the proof is the construction of a function w which represents the oscillating part of a solution u and reduces the problem (1) to an inhomogeneous heat equation for the smooth part v of the solution $u = v + w$.

We describe an idea of G. Strang [6] for constructing w which simplifies the original argument in [5]. Consider the following (infinite) triangulation of $[0,1]^2$

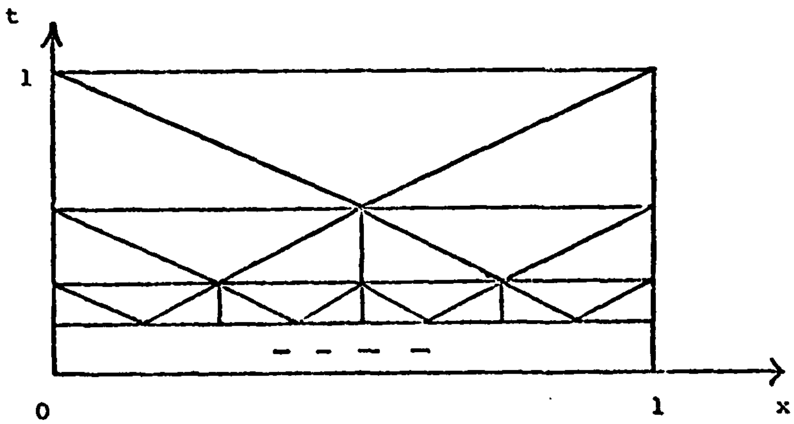


Figure 3

which is determined by the points

$$z_{jk} := (k2^{-j}, 2^{-j}), \quad k = 0,\dots,2^j, \quad j \in \mathbb{N} .$$

Set $h(0) = 0$, $h'(x) = \max(0, f'(x) - 1)$. Define w as the piecewise linear function with respect to this partition which interpolates the data

$$w_{jk} := \max\{v2^{-j} : v2^{-j} \leq h(k2^{-j}),\ v \in \mathbb{Z}\}$$

at the points z_{jk}.

By the definition of w_{jk},

$$w_{jk} \leq h(k2^{-j}) < w_{jk} + 2^{-j} ,$$

and since $0 \leq h' \leq 1$ we have

$$w_{j(k+1)} - w_{jk} \in \{0, 2^{-j}\} .$$

One then easily verifies that

(i) $w_x : [0,1] \times [0,T] \to \{0,1\}$

(ii) $w_t \in L^\infty$

(iii) $\lim\limits_{t \to 0} \|w(\cdot,t) - h\|_\infty = 0$ (4)

(iv) $f'(x) \leq 1 - 2^{-j} \implies w_x(x,t) = 0$ for $t \leq 2^{-j}$.

Property (iv) means that for small t $w_x(\cdot,t)$ is supported in a neighborhood of the set $\{x : f'(x) \geq 1\}$. Let v be the solution of the problem

$$v_t + w_t = v_{xx}$$
$$v_x(0,t) = v_x(1,t) = 0 \tag{5}$$
$$v(x,0) = f(x) - h(x) .$$

We claim that $u = v + w$ is a solution of problem (P).

Since $w_t \in L^\infty$, v_x is Hölder continuous and $v_t \in L^p$ for any $p < \infty$ by standard estimates for the heat equation. Put $\kappa := \min(a - 1, 2 - b)$. By the continuity of v_x and (4(iv)) there exists $T > 0$ such that

$$v_x(x,t) \leq 1 + \kappa, \quad (x,t) \in [0,1] \times [0,T] ,$$

and

$$(\mathrm{supp}\ w_x) \cap [0,1] \times [0,T] \subseteq$$
$$\{(x,t) : t \in [0,T], v_x(x,t) > 1 - \kappa\} .$$

In view of the relation

$$\phi(\xi + 1) = \phi(\xi), \ \xi \in [1 - \kappa, 1 + \kappa] , \tag{6}$$

it follows that

$$v_t + w_t = v_{xx} = \phi(v_x)_x = \phi(v_x + w_x)_x .$$

Clearly, the function w is not uniquely determined. For any $\lambda > 0$, $w_\lambda(x,t) := w(x,\lambda t)$ satisfies (4). Since v_x is continuous the solutions $u_\lambda = v_\lambda + w_\lambda$ are distinguished by their different discontinuity patterns. Thus there is a continuum of solutions to the problem (P).

If the monotone increasing parts of ϕ have different slopes the proof of the Theorem is considerably more complicated (see [5]). The reason is that instead of (6) we have the relation

$$\phi(\xi + (A\xi + B)) = \phi(\xi) , \tag{7}$$

where the shift is no longer independent of ξ. Nevertheless, the proof is based on the construction of an auxiliary function w which, however, depends on v and the equation for the smooth part v becomes quasilinear. For a general smooth nonmonotone ϕ the existence question is not settled.

In spite of nonuniqueness, numerical approximations to the problem (P) are stable; the x-derivatives of the numerical solution have the oscillatory behavior asserted by the Theorem.

We used a Crank-Nicholson scheme

$$u_j^{n+1} - u_j^n = \frac{\Delta t}{\Delta x} D\phi\Big(\frac{\Delta x}{2}\,(Du_{j-1}^{n+1} + Du_{j-1}^n)\Big) \tag{8}$$

where u_j^n approximates $u(j\Delta x, n\Delta t)$ and $Du_j = u_{j+1} - u_j$ denotes the forward difference. For ϕ in Figure 2 with $a = 1.25$, $b = 1.75$ and the model initial data $f(x) = 3x^2 - 2x^3$ the numerical solutions show the following typical behavior. For small t, $u\Delta t < 0.1$, Du_j^n oscillates over the interval where $\phi'(f')$ is negative as shown in figure (4) below

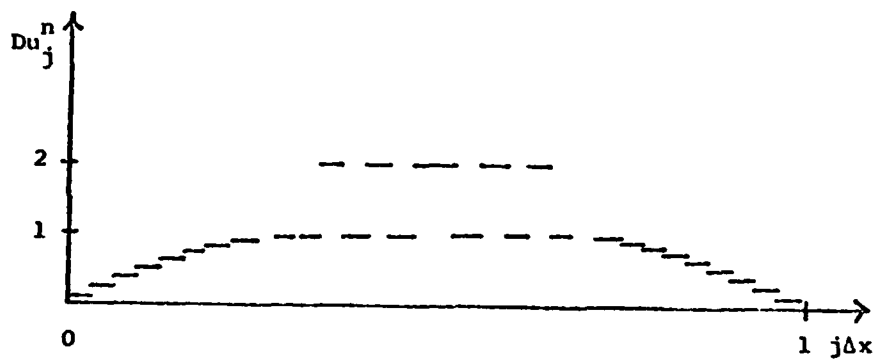


Figure 4

where $\Delta x = 2^{-5}$, $\Delta t = (\Delta x)^2$, $n\Delta t = 2^{-7}$. The oscillations are very regular. Du_j^n alternates between the values 1 and 2 which, in agreement with the Theorem, differ by 1 and do not lie in the interval $[1.25, 1.75]$ where ϕ' is negative. Moreover, $\phi(Du_j^n)$ is (numerically) continuous.

As $t = n\Delta t$ increases the oscillations gradually disappear and u_j^n approaches the limiting state $u_x = 0$. Similar numerical approximations have been independently studied by G. Strang and A. Naby [6]. It has not been established that the numerical approximations converge and thereby possibly single out a distinguished solution of the problem (P).

4. A RELATED CONVEXIFIED PROBLEM

It is known [1] that in one space dimension the solution of the problem

$$0 = \phi(u_x)_x, \qquad x \in [0,1]$$
$$u(0) = 0, \quad u(1) = d, \quad d \text{ a constant }, \tag{9}$$

can be obtained by minimizing the functional

$$J(u) = \int_0^1 \Phi(u_x(x))dx, \qquad \Phi'(\cdot) = \phi(\cdot) ,$$

over all functions $u \in W^{1,2}([0,1])$ satisfying the given boundary conditions, under suitable assumptions on Φ (these do not require Φ convex). If ϕ satisfies assumptions (A), (9) is the steady state problem associated with (P). By a result of Ekeland and Témam [4] such a solution u of the variational problem also *minimizes* the functional

$$\bar{J}(u) = \int_0^1 \bar{\Phi}(u_x(x))dx ,$$

where $\bar{\Phi}$ is the convexification of Φ, and u solves the convexified problem:

$$0 = \bar{\phi}(u_x)_x, \qquad x \in [0,1]$$
$$u(0) = 0, \quad u(1) = d, \quad \bar{\phi}(\cdot) = \bar{\Phi}'(\cdot) ; \tag{10}$$

the function $\bar{\phi}(\cdot)$ is necessarily nondecreasing on $\mathbb{R}$ (if ϕ has the graph in Figure 2, Figure 5 below is the graph of $\bar{\phi}$). An analogous variational principle has been established for problem (P), but only for ϕ monotone (see [2]).

Motivated by these results we consider the convexified problem associated with (P):

$$u_t = \bar{\phi}(u_x)_x, \qquad (x,t) \in [0,1] \times [0,T] ,$$
$$u_x(0,t) = u_x(1,t) = 0, \quad u(x,0) = f(x) . \tag{$\bar{P}$}$$

By the Theorem below, solutions of (P) and ($\bar{P}$) show a qualitatively different behavior and, at first sight, there is no obvious connection between them when the data f' is smooth and falls in the critical range (a,b). Nevertheless, it appears plausible that as $t \to \infty$ problems (P) and ($\bar{P}$) approach the same steady state.

For simplicity we discuss ($\bar{P}$) only for the piecewise linear constitutive function considered in Section 3. For ϕ in Figure 2, $\bar{\phi}$ is of the form

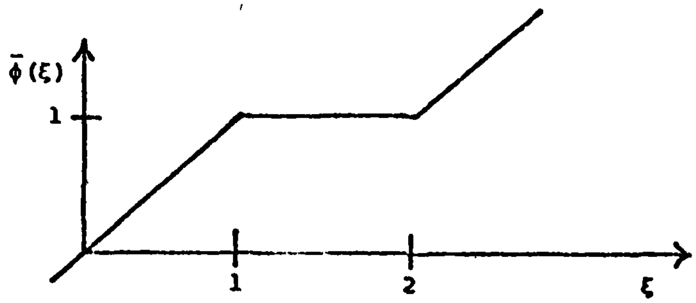


Figure 5

The graph of $\bar{\phi}$ is familiar from the Stefan problem. In fact, if the initial data satisfy

$$f'(x) \notin (1,2), \quad x \in [0,1] ,$$
$$\bar{\phi}(f') \ \text{continuous} ,$$

($\bar{P}$) describes a Stefan problem. If, however, f is smooth which we assume throughout this section, the evolution of the free boundaries resulting from ($\bar{P}$) is different. For simplicity we assume that

$$\begin{aligned} & f'(x) < 2, \quad x \in [0,1] , \\ & \{x : f'(x) > 1\} = (r_0,s_0), \quad 0 < r_0 < s_0 < 1 , \qquad (11) \\ & f''(r_0) > 0, \quad f''(s_0) < 0 . \end{aligned}$$

Theorem. Problem ($\bar{P}$) with $\bar{\phi}$ given in Figure 5 has a unique solution u on $[0,1] \times \mathbb{R}_+$ with

$$\|u_x\|_\infty < \|f'\|_\infty$$

$$\|u_t\|_\infty \leq \|\phi(f')'\|_\infty .$$

If f satisfies assumptions (11), the region $\Omega := \{(x,t) : u_x(x,t) > 1\}$ where $u_t = 0$ is bounded by two curves $r,s \in H^{1/2}$ (Hölder class) connecting the points $(r_0,0),(s_0,0)$ with the point (x_*,T_*) (Figure 6). Moreover,

$$T_* \leq \|f'\|_\infty . \tag{12}$$

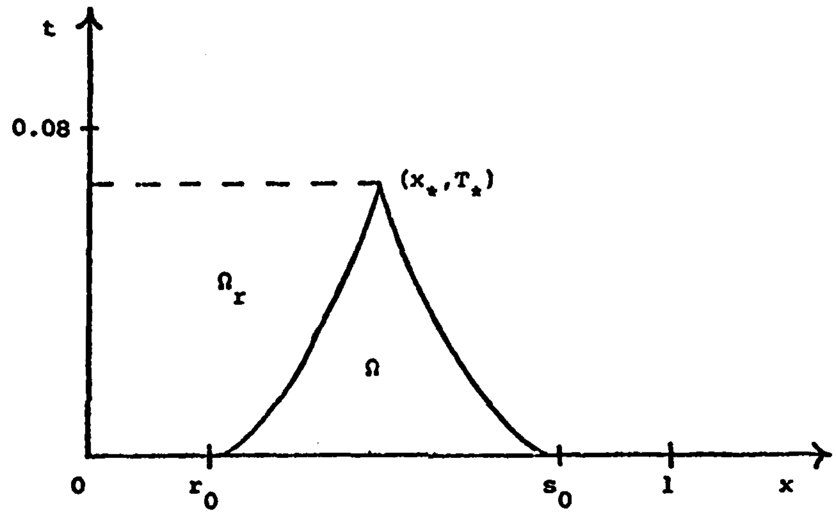


Figure 6

Note that u satisfies the heat equation on $[0,1] \times \mathbb{R}_+ \setminus \Omega$ and the regularity assertions in particular describe the behavior of u in a neighborhood of the points $(r_0,0),(s_0,0)$.

The regularity for u,r,s is proved by using the implicit semidiscrete approximation

$$u(x,(n+1)\Delta t) - u(x,n\Delta t) = \Delta t\phi(u_x(x,(n+1)\Delta t))_x, \; x \in [0,1] ,$$

of the equation $u_t = \bar{\phi}(u_x)_x$. We prove the inequality (12). Since u is continuous and $u(x,t) = u(x,0)$ on Ω we have

$$u(r(t),t) = f(r(t)) . \tag{13}$$

Also, note that

$$u_x(r(t)^-,t) = 1 . \tag{14}$$

Integrating the equation $u_t = \bar{\phi}(u_x)_x$ over the domain $\Omega_r := \{(x,t) : 0 < x < r(t),\ t \in (0,T_*)\}$ we obtain

$$0 = \iint_{\Omega_r} (u_t - u_{xx})\,dxdt =$$

$$- \int_0^{r_0} f(x)dx - \int_0^{T_*} f(r(t))\dot{r}(t)dt + \int_0^{r(T_*)} u(x,T_*)dx - \int_0^{T_*} 1\ dt\ .$$

Using (13) and $\|u_x\|_\infty \le \|f'\|_\infty$ for the estimate, it follows that

$$T_* = \int_0^{r(T_*)} (u(x,T_*) - f(x))dx \le \int_0^1 \int_x^{r(T_*)} 2\|f'\|_\infty dydx \le \|f'\|_\infty\ .$$

Numerical computations show (Figure 6 is a particular example with $f(x) = 3x^2 - 2x^3$) that the curves r,s touch the x-axis. Formally, this can be explained as follows. Differentiating equation (13) with respect to t and using (14) we have

$$\dot{r}(t)(f'(r(t)) - 1) = u_t(r(t),t)\ .$$

Since $f'(r(0)) = f'(r_0) = 1$ we can rewrite this in the form

$$\dot{r}(t)(r(t) - r_0) = u_t(r(t),t)/g(r(t) - r_0)$$

where $g(\xi - r_0) := \dfrac{f'(\xi) - f'(r_0)}{\xi - r_0}$. If we assume that u_t, u_{xx} are continuous on $\overline{\Omega_r}$ it follows that

$$r(t) = r_0 + \sqrt{2t} + o(\sqrt{t}) \quad (t \to 0)\ , \tag{15}$$

which is consistent with regularity of r established in the theorem above.

We have not yet succeeded in proving (15). The difficulty is that the free boundary $x = r(t)$ is characteristic for $(\bar{P})$ at the point $(r_0,0)$.

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We discuss the well-posedness of the model problem

$$u_t = \phi(u_x)_x \quad \text{on} \quad [0,1] \times [0,T],\ T > 0\ , \qquad (P)$$

subject to given Neumann or Dirichlet boundary conditions at $x = 0$ and $x = 1$, and to the initial condition $u(x,0) = f(x)$; the given functions $f : [0,1] \to \mathbb{R}$, $\phi : \mathbb{R} \to \mathbb{R}$ are assumed to be smooth, $\phi(0) = 0$, ϕ satisfies

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20. ABSTRACT - cont'd.

the coercivity assumption $\xi\phi(\xi) \geq c\xi^2$, for some constant $c > 0$ and for $\xi \in \mathbb{R}$, and ϕ is assumed to be decreasing on an interval (a,b) with $a > 0$. We present a recent nonuniqueness result in the special case when ϕ is piecewise linear and study a related convexified problem.